

Simplicity over Singular Hyperbolicity

Mohammad Fanaee

March 25, 2014

Abstract

We show that Lyapunov exponents of typical Hölder continuous fiber bunched linear cocycles over singular hyperbolic flows have multiplicity 1: the subspace of exceptional cocycles has infinite codimension.

Contents

1	Introduction	1
2	Cocycles over Poincaré Return Maps	7
3	The Unique physical probability measure	7
4	Typical Simplicity	8
	References	9

1 Introduction

Let

$$f^t : M \rightarrow M$$

be a C^1 flow on a measurable space M , and μ a probability measure on M which is invariant by f^t . Assume that $\pi : \mathcal{V} \rightarrow M$ a d -dimensional vector bundle over M endowed with a measurable Riemannian metric. A linear cocycle over f^t is a flow

$$F^t : \mathcal{V} \rightarrow \mathcal{V}$$

which acts by linear isomorphisms $F_x^t : \mathcal{V}_x \rightarrow \mathcal{V}_{f^t(x)}$ on the fibers; i.e.

- (i) $F_x^0 = id$ and
 - (ii) $F_x^{t+s} = F_{f^s(x)}^t F_x^s$, for any $t, s \in \mathbb{R}$.
- If

$$x \mapsto \max\{0, \log \|F_x^1\|\}$$

is μ -integrable then, by Oseledets Theorem [O69], there exist a decomposition (Lyapunov splitting)

$$\mathcal{V}_x = E_1(x) \oplus \dots \oplus E_k(x), \quad 1 \leq k = k(x) \leq d$$

of the vector bundle and real numbers (Lyapunov exponents) $\lambda_1 > \dots > \lambda_k$,

$$\lambda_i = \lim_{|t| \rightarrow +\infty} \frac{1}{t} \log \|F_x^t(v)\|, \quad 1 \leq i \leq k.$$

The Lyapunov splitting and the Lyapunov exponents are invariant by f^t and vary measurably with the base point x . Then, the Lyapunov splitting and Lyapunov exponents are constant if μ is an ergodic probability measure.

One interesting problem is to study the conditions under which these exponents have multiplicity 1 ($k = d$). We will call such a cocycle as a “simple cocycle”.

Bonatti and Viana [BV04] proved some criterion for simple cocycles over shift maps of finite type. This is improved by Avila and Viana in [AV07] for cocycles over any Markov map, whom used to prove the Zorich-Kontsevich conjecture in [AV07]. In [F14] we prove that Avila and Viana criterion is typical in the space of all Hölder continuous fiber bunched linear cocycles over Markov maps. In all of these results, a local product structure is assumed for invariant probability measures.

Here, we study simple cocycles over flows, in particular, simple cocycles over singular hyperbolic flows. Singular hyperbolic sets are motivated by extension of the classical construction of Lorenz-like flows [W79, GW79]; in a robust way by regular orbits. It is shown in [MPP04] that any attractor of a 3-dimensional flow containing in a robust fashion equilibria together with regular orbits must be singular hyperbolic, that is, it must admit an invariant splitting

$$E^s \oplus E^{cu}$$

of the tangent bundle into a 1-dimensional uniformly contracting sub-bundle and a 2-dimensional volume-expanding sub-bundle.

Some of the topologic and ergodic features of hyperbolic systems were already extended for this class of flows. Indeed, we are able to construct convenient cross-sections and invariant contracting foliations for a corresponding Poincaré map P , and so its geometric and ergodic properties can be well understood. It is proved, for example, in [APPV09] that there exists a unique invariant probability measure for singular hyperbolic flows which is physical.

Here, we prove that simple cocycles are typical in the space of all Hölder continuous fiber bunched linear cocycles over a singular hyperbolic flow. By typical, we mean that the set of exceptional cocycles has infinite codimension, i.e. it is contained in a finite union of closed submanifolds with arbitrary high codimension.

Since the flow of the Lorenz equation [L63] and the Lorenz-like flows [W79, GW79] display singular hyperbolic attractors, our result is valid, in particular, for linear cocycles over these flows.

1.1 Singular hyperbolicity

In [MPP04] they proved that any C^1 robustly transitive sets with a finite number of singularities on closed 3-manifolds verifies a weaker form of hyperbolicity meaning that

- (i) it is either an attractor or a repeller,
- (ii) eigenvalues on any singularity satisfy the same inequalities as in the Lorenz-like model

$$\alpha_{ss} < \alpha_s < 0 < -\alpha_s < \alpha_u$$

and

- (iii) they are partially hyperbolic and the central direction is volume expanding.

There exists a finite number of singularities and then one may consider a finite number of Poincaré return maps of the flow to convenient cross-sections, with respect to any singularity. That is, continuous maps

$$P : \Sigma \rightarrow \Sigma'$$

as

$$P(x) = f^{r(x)}(x)$$

where $r(\cdot)$ is the Poincaré return time map.

1.2 Simplicity criterion

Let $f : N \rightarrow N$ be an invertible and measurable Markov map. A linear cocycle over f is a transformation

$$F_A : N \times \mathbb{C}^d \rightarrow N \times \mathbb{C}^d$$

satisfying $f \circ \pi = \pi \circ F_A$ which acts by linear isomorphisms $A(x)$ on fibers. So, the cocycle has the form

$$F_A(x, v) = (f(x), A(x)v)$$

where $A : N \rightarrow \text{GL}(d, \mathbb{C})$, and $A^0(x) = \text{id}$. By definition $F_A^n(x, v) = (f^n(x), A^n(x)v)$, where

$$A^n(x) = A(f^{n-1}(x)) \dots A(f(x))A(x),$$

$$A^{-n}(x) = (A^n(f^{-n}(x)))^{-1},$$

for any $n \in \mathbb{N}$.

Let a probability measure ν invariant by f . If $x \mapsto \max\{0, \log \|A(x)\|\}$ is μ -integrable then the Oseledets Theorem [O68] states that there exist an invariant Lyapunov splitting

$$E_1(x) \oplus \dots \oplus E_k(x), \quad 1 \leq k = k(x) \leq d,$$

of \mathbb{C}^d and Lyapunov exponents $\lambda_1(x) > \dots > \lambda_k(x)$,

$$\lambda_i(x) = \lim_{|n| \rightarrow +\infty} \frac{1}{n} \log \|A^n(x)v_i\|, \quad v_i \in E_i(x), \quad 1 \leq i \leq k,$$

at ν -almost every point.

Lyapunov exponents are invariant, uniquely defined at almost every x and vary measurably with the base point x . Thus, Lyapunov exponents are constant, almost every where, when the invariant measure is ergodic: $\{\lambda_1, \dots, \lambda_k\}$.

Suppose that N is endowed with a metric d for which

- (i) $d(f(y), f(z)) \leq \theta(x)d(y, z)$, for all $y, z \in W_{\text{loc}}^s(x)$,
 - (ii) $d(f^{-1}(y), f^{-1}(z)) \leq \theta(x)d(y, z)$, for all $y, z \in W_{\text{loc}}^u(x)$,
- where $0 < \theta(x) \leq \theta < 1$, for all $x \in N$.

Let F_A be an η -Hölder continuous linear cocycle over f .

Definition 1.1 F_A is fiber bunched if there exists some constant $\tau \in (0, 1)$ such that

$$\|A(x)\| \|A(x)^{-1}\| \theta(x)^\eta < \tau,$$

for any $x \in N$.

Let $H_{x,y}^n = A^n(y)^{-1}A^n(x)$.

Definition 1.2 A cocycle A admits s -holonomy if

$$H_{x,y}^s = \lim_{n \rightarrow +\infty} H_{x,y}^n$$

exists for any pair of points x, y in the same local stable set. u -holonomy is defined in a similar way, when $n \rightarrow -\infty$, for pairs of points in the same local unstable set.

Remark 1.1 One may verify that, for all x and any $y \in W_{\text{loc}}^{s(u)}(x)$,

(i) $H_{x,y}^{s(u)} = H_{z,y}^{s(u)} \cdot H_{x,z}^{s(u)}$, for any $z \in W_{\text{loc}}^{s(u)}(x)$, and $H_{y,x}^{s(u)} \cdot H_{x,y}^{s(u)} = \text{id}$,

(ii) $H_{f^j(x), f^j(y)}^{s(u)} = A^j(y) \circ H_{x,y}^s \circ A^j(x)^{-1}$, for all $j \geq 1$.

For more details see [4, 10].

1.3 Local product structure

Let $N_u = \mathbb{N}^{n \geq 0}$ and $N_s = \mathbb{N}^{n < 0}$. The map

$$x \mapsto (x_s, x_u)$$

is a homeomorphism from N onto $N_s \times N_u$ where $x_s = \pi_s(x)$ and $x_u = \pi_u(x)$, for natural projections $\pi_s : N \rightarrow N_s$ and $\pi_u : N \rightarrow N_u$. We also consider the maps $f_s : N_s \rightarrow N_s$ and $f_u : N_u \rightarrow N_u$ defined by

$$f_u \circ \pi_u = \pi_u \circ f,$$

$$f_s \circ \pi_s = \pi_s \circ f^{-1}.$$

Assume that μ_f is an ergodic probability measure for f . Let $\mu_s = (\pi_s)_* \mu_f$ and $\mu_u = (\pi_u)_* \mu_f$ be the images of μ_f under the natural projections. It is easy to see that μ_s and μ_u are ergodic probabilities for f_s and f_u , respectively. Notice that μ_s and μ_u are positive on cylinders, by definition.

We say that μ_f has local product structure if there exists a measurable density function $\omega : N \rightarrow (0, +\infty)$ such that

$$\mu_f = \omega(x)(\mu_s \times \mu_u).$$

1.4 Simple cocycles

Let (f, μ) an ergodic complete shift map where μ has product structure and let A be a linear cocycle over f . Suppose that p is a periodic point of f , and q a homoclinic point of p , i.e. $q \in W_{\text{loc}}^u(p)$ and there is some multiple $m \geq 1$ of $\text{per}(p)$ such that $f^m(q) \in W_{\text{loc}}^s(p)$. We define the *transition map*

$$\Psi_{A,p,q} : \mathbb{C}_p^d \rightarrow \mathbb{C}_p^d$$

by

$$\Psi_{A,p,q} = H_{f^m(q),p}^s A^m(q) H_{p,q}^u \in \text{GL}(d, \mathbb{C}).$$

Definition 1.3 *A is pinching at p if all eigenvalues of $A^{\text{per}(p)}(p)$ have distinct absolute values. A is twisting at p, q if, for any pair of invariant subspaces E_1, E_2 of $A^{\text{per}(p)}(p)$ with $\dim E_1 + \dim E_2 = d$,*

$$\Psi_{A,p,q}(E_1) \cap E_2 = \{\mathbf{0}\}.$$

A cocycle A is simple if there exist some periodic point p and some homoclinic point q of p such that A is pinching at p and twisting at p, q.

Theorem 1.1 [AV07] *If A is simple then Lyapunov spectrum of A is simple.*

Assume that A is a fiber bunched linear cocycle over complete shift map. Then s-holonomy and u-holonomy exist. Moreover, the holonomy map $H_{x,y}^{s(u)}$ vary continuously on (x, y) in the sense that the map

$$(x, y) \rightarrow H_{x,y}^s$$

is continuous on $W_n^s = \{(x, y) : f^n(y) \in W_{\text{loc}}^s(x)\}$, for every $n \geq 0$. It is, in fact, a direct consequence of the uniform limit of the last definition when $(x, y) \in W_0^s$, for instance. The general case $n > 0$ follows immediately, by (ii) of the last remark. Therefore, one may consider this notion of dependence:

$$(A, x, y) \rightarrow H_{A,x,y}^s$$

is continuous on $C^{r,\rho}(M, d, \mathbb{C}) \times W_n^s$, for all $n \geq 0$.

Even more, the map

$$B \mapsto H_{B,x,y}^{s(u)}$$

is of class C^1 on an open set \mathcal{U} , for any $y \in W_{\text{loc}}^{s(u)}(x)$. Notice that $\mathcal{C}^{r,\rho}(N, d, \mathbb{C})$ is the Banach space of all $C^{r,\rho}$ maps from N to the space of all $d \times d$ invertible matrices, and so the tangent space at each point $A \in \mathcal{C}^{r,\rho}(N, d, \mathbb{C})$ is naturally identified with that Banach space.

Then, by rather geometric submersion tools we show that pinching and twisting are typical conditions in the space of fiber bunched cocycles and so, we conclude that

Theorem 1.2 (F14) *Simple cocycles are typical in the space of all fiber bunched linear cocycles over any Markov map.*

1.5 Suspension flows

Consider a suspension flow $f^t : M \rightarrow M$ of $f : N \rightarrow N$ and let $T : N \rightarrow \mathbb{R}$ be the corresponding first return time map to N . Assume that $A^t : M \rightarrow \text{GL}(d, \mathbb{C})$ is a linear cocycle over f^t . We define

$$A_f(x) = A^{T(x)}(x),$$

for any $x \in N$. Then $A_f : N \rightarrow \text{GL}(d, \mathbb{C})$ is a linear cocycle over f .

Let, for any $r \in \mathbb{N} \cup \{0\}$ and $0 \leq \rho \leq 1$ with $r + \rho > 0$, the Banach space

$$\mathcal{C}^{r,\rho}(M, d, \mathbb{C}) = \{A^t : M \rightarrow \text{GL}(d, \mathbb{C}) : \|A_f\|_{r,\rho} < +\infty\}.$$

It turns out that A^t is η -Hölder continuous if the corresponding (discrete time) linear cocycle A_f is Hölder.

Recall that the $C^{r,\rho}$ topology is defined by

$$\|A\|_{r,\rho} = \max_{0 \leq i \leq r} \sup_x \|D^i A(x)\| + \sup_{x \neq y} \frac{\|D^r A(x) - D^r A(y)\|}{d(x, y)^\rho}$$

(for $\rho = 0$ omit the last term) and then

$$\mathcal{C}^{r,\rho}(N, d, \mathbb{C}) = \{A : N \rightarrow \text{GL}(d, \mathbb{C}) : \|A\|_{r,\rho} < +\infty\}$$

is a Banach space. We assume that $r + \rho > 0$ which implies η -Hölder continuity:

$$\|A(x) - A(y)\| \leq \|A\|_{0,\eta} d(x, y)^\eta,$$

with

$$\eta = \begin{cases} \rho & r = 0 \\ 1 & r \geq 1. \end{cases}$$

Now, let A^t be an η -Hölder continuous linear cocycle over f^t .

Definition 1.4 *A^t is fiber bunched if the corresponding linear cocycle A_f is a fiber bunched linear cocycle over f .*

Note that fiber bunching is also an open condition in $\mathcal{C}^{r,\rho}(\Lambda, d, \mathbb{C})$, by definition.

2 Cocycles over Poincaré Return Maps

2.1 Markov structure on cross sections

Theorem 2.1 (AP07) *There is a system of transversal sections Σ such that for any band $B \subset \Sigma$ there is a transition map $T : B \rightarrow \Sigma$ that $T(B)$ covers Σ_+ or Σ_- .*

3 The Unique physical probability measure

Theorem 3.1 (APPV08) *Λ supports a unique physical invariant probability measure μ which is ergodic and its ergodic basin covers a full Lebesgue measure subset of the topological basin of attractor.*

3.1 Local product structure of the probability measure

Proposition 3.1 *The lift probability μ has product structure. Moreover, the density function ω is continuous and, bounded from zero and infinity*

Proof. Note that by Theorem 3.1, for all \hat{x}, \hat{y} in the same cylinder

$$\log \frac{J\hat{f}(\hat{x})}{J\hat{f}(\hat{y})} \leq c^{n(x,y)}.$$

The rest of proof is based on 4 main steps *Step 1.* If $\hat{x}, \hat{y} \in \hat{N}$ then for any $x \in W_{\text{loc}}^s(\hat{x})$ and $y \in W_{\text{loc}}^u(x) \cap W_{\text{loc}}^s(\hat{y})$, the limit

$$J_{\hat{x}, \hat{y}}(x) = \lim_{n \rightarrow \infty} \frac{J\hat{f}^n(\hat{x}^n)}{J\hat{f}^n(\hat{y}^n)},$$

where $\hat{x}^n = \hat{\pi}(f^{-n}(x))$, $\hat{y}^n = \hat{\pi}(f^{-n}(y))$, exists uniformly on \hat{x}, \hat{y}, x . Moreover,

$$(\hat{x}, \hat{y}, x) \mapsto J_{\hat{x}, \hat{y}}(x)$$

is continuous and uniformly bounded from zero and infinity.

Indeed, we observe that

$$\log \frac{J\hat{f}^n(\hat{x}^n)}{J\hat{f}^n(\hat{y}^n)} \leq \sum_{i=1}^n \log \frac{J\hat{f}(\hat{x}^i)}{J\hat{f}(\hat{y}^i)}.$$

Since \hat{x}^i and \hat{y}^i are in the same cylinder, the series is uniformly bounded by $\sum_i c^{n(\hat{x}^i, \hat{y}^i)}$. But $n(\hat{x}^i, \hat{y}^i)$ is strictly increasing that implies uniform convergence of the series.

Step 2. If $\{\mu_{\hat{x}} : \hat{x} \in \hat{N}\}$ be an integration of μ then, for μ -almost every $\hat{x} \in \hat{N}$,

$$\mu_{\hat{x}}(\xi_n) = \frac{1}{J\hat{f}^n(\hat{x}^n)},$$

for every cylinder $\xi_n = [x_{-n}, \dots, x_{-1}]$, $n \geq 1$, and any $x \in \xi_n \times \{\hat{x}\}$.

Step 3. Given any disintegration, by the last step, one may find a disintegration $\{\mu_{\hat{x}} : \hat{x} \in \hat{N}\}$ of μ so that

$$\mu_{\hat{y}} = J_{\hat{x}, \hat{y}} \mu_{\hat{x}}.$$

Step 4. Fixing any $\hat{x}_0 \in \hat{N}$, one may define

$$\hat{\omega}(x_s, x_u) = J_{\hat{x}_0, x_u}(x_s, x_u),$$

for every $x = (x_s, x_u) \in N$. By Step 2, $\mu_{x_u} = \hat{\omega}(x_s, x_u)$, for any $x_u \in \hat{N}$.

The lift measure μ projects to $\hat{\mu} = \mu_u$, but the projection μ_s to N_s is given by

$$\mu_s = \mu_{\hat{x}_0} \int_{\hat{N}} \hat{\omega}(x_s, x_u) d\hat{\mu}.$$

Therefore

$$\mu = \omega(x_s, x_u) \mu_s \times \mu_u$$

where

$$\omega(x_s, x_u) = \frac{1}{\int_{\hat{N}} \hat{\omega}(x_s, x_u) d\hat{\mu}} \hat{\omega}(x_s, x_u).$$

As conditional probabilities vary continuously with the base point so the density function ω is continuous. Also, ω is bounded from zero and infinity.

The the proof of Proposition 3.1 is now completed.

4 Typical Simplicity

Now, we are in the setting to complete the proof of Main Theorem.

For any linear cocycle A^t over Λ consider the corresponding linear cocycle A_f on N by

$$A_f(x) = A^{T(x)}(x),$$

for any $x \in N$.

Proposition 4.1 *Lyapunov spectrum of A^t is simple if and only if Lyapunov spectrum of A_f is simple.*

Proof. The Lyapunov exponents of A_f are obtained by multiplying those of A^t by the average return time

$$s_n(x) = \sum_{j=0}^{n-1} T(\hat{P}^j(x)), \quad x \in N.$$

Indeed, given any non zero vector v ,

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log \|A_f^n(x)v\| = \lim_{n \rightarrow +\infty} \frac{1}{n} \log \|A^{s_n(x)}(x)v\|$$

which, for μ -almost every x , this is equal to

$$\lim_{n \rightarrow +\infty} \frac{1}{n} s_n(x) \lim_{m \rightarrow +\infty} \frac{1}{m} \log \|A^m(x)v\|.$$

But $\frac{1}{n} s_n(x)$ converges to $\int T \, d\mu < +\infty$

The proof of Proposition 4.1 is now completed.

Let A^t be a linear cocycle over Λ . We define a neighborhood \mathcal{V} of A^t as the subset of all cocycles B^t over Λ for which $B_f \in \mathcal{U}$.

Proposition 4.2 *The application*

$$\mathcal{V} \ni B^t \mapsto B_f \in \mathcal{U}$$

is a submersion.

Proof. By definition,

$$\partial_{B^t} B_f(\dot{B}_t) = \dot{B}_f.$$

Let $\dot{B} \in \mathcal{C}^{r,\rho}(N, d, \mathbb{C})$. Then the suspension \dot{B}^t of \dot{B} is defined by

$$\dot{B}^t(X^s(x)) = (\text{id}, t+s), \quad 0 < t+s \leq T(x),$$

identifying $(\text{id}, T(x))$ with $(\dot{B}(x), 0)$, for any $x \in N$, setting $\dot{B}^0 = \text{id}$. \dot{B}^t is an η -Hölder linear cocycle over Λ for which $\dot{B}_f(x) = (\dot{B}(x), 0)$. This shows that the derivative is surjective.

The proof of Proposition 4.2 is now completed.

The proof of Main Theorem is then completed, by Theorem 1.1.

References

- [AP10] V. Araujo and M. Pacifico. Three dimensional flows. Springer-Verlag (2010).
- [AP07] A. Arroyo and E. Pujals. Dynamical properties of singular-hyperbolic attractors. Discrete and Continuous Dynamical Systems 19 (2007) 67–87.
- [APPV09] V. Araujo, M. Pacifico, E. Pujals and M. Viana. Singular-hyperbolic attractors. Transactions of the American Mathematical Society 361 (2009) 2431-2485.
- [ASV14] A. Avila, J. Santamaria and M. Viana. Holonomy invariance: rough regularity and applications to Lyapunov exponents. preprint IMPA.
- [AV07] A. Avila and M. Viana. Simplicity of Lyapunov spectra: a sufficient condition. Portugaliae Mathematica 64 (2007) 311-376.

- [AV07] A. Avila and M. Viana.
- [BDV04] C. Bonatti, L. Diaz and M. Viana. Dynamics Beyond Uniform Hyperbolicity: A Global Geometric and Probabilistic Perspective. Encyclopaedia of Mathematical Sciences 102 Springer-Verlag (2004).
- [BGV03] C. Bonatti, X. Gomez-Mont and M. Viana. Généricité d'exposants de Lyapunov non-nuls pour des produits déterministes de matrices. Annales de l'Institut Henri Poincaré 20 (2003) 579-624.
- [BV04] C. Bonatti and M. Viana. Lyapunov exponents with multiplicity 1 for deterministic products of matrices. Ergodic Theory and Dynamical Systems 24 (2004) 1295-1330.
- [F13] M. Fanaee. Linear cocycles over Lorenz-like flows. Publicaciones Matemáticas del Uruguay 14 (2013) 136-146.
- [F14] M. Fanaee. On simple cocycles over Markov maps.
- [GW79] J. Guckenheimer and R. Williams. Structural stability of Lorenz attractors. Publications Mathématiques de l'IHÉS 50 (1979) 59-72.
- [L86] F. Ledrappier. Positivity of the exponent for stationary sequences of matrices, Lectur Notes in Mathematics Springer 1186 (1986) 5673.
- [L63] E. Lorenz. Deterministic nonperiodic flow. Journal of the Atmospheric Sciences 20 (1963) 130-141.
- [MPP99] C. Morales, M. Pacico and E. Pujals. On singular hyperbolic systems. Proceedings of the American Mathematical Society 127 (1999) 3393-3400.
- [M04] C. Morales, M. Pacifico and E. Pujals. Robust transitive singular sets for 3-flows are partially hyperbolic attractors or repellers. Annals of Mathematics 160 (2004) 357-432.
- [O68] V. Oseledec. A multiplicative ergodic theorem. Transactions of the Moscow Mathematical Society 19 (1968) 197-231.
- [T98] W. Tucker. The Lorenz attractor exists. University of Uppsala (1998).
- [V08] M. Viana, Almost all cocycles over any hyperbolic system have non-vanishing Lyapunov exponents, Annals of Mathematics 167 (2008) 643-680.
- [W79] R. Williams, The structure of the Lorenz attractor, Publications Mathématiques de l'IHÉS 50 (1979) 73-99.

Mohammad Fanaee
Instituto de Matemática e Estatística (IME)
Universidade Federal Fluminense (UFF)
Campus Valonguinho
24.020-140 Niterói, RJ - Brazil
Email: mf@id.uff.br